

ON THE RANK OF THE 2-CLASS GROUP OF $\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-1})$

ABDELMALEK AZIZI, MOHAMMED TAOUS, AND ABDELKADER ZEKHNINI

ABSTRACT. Let d be a square-free integer, $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$ and $i = \sqrt{-1}$. Let $\mathbf{k}_1^{(2)}$ be the Hilbert 2-class field of \mathbf{k} , $\mathbf{k}_2^{(2)}$ be the Hilbert 2-class field of $\mathbf{k}_1^{(2)}$ and $G = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ be the Galois group of $\mathbf{k}_2^{(2)}/\mathbf{k}$. Our goal is to give necessary and sufficient conditions to have G metacyclic in the case where $d = pq$, with p and q are primes such that $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$ or $p \equiv 1 \pmod{8}$ and $q \equiv 3 \pmod{4}$.

1. INTRODUCTION

Let k be an algebraic number field and let $Cl_2(k)$ denote its 2-class group. Denote by $k_2^{(1)}$ the Hilbert 2-class field of k and by $k_2^{(2)}$ its second Hilbert 2-class field. Put $G = \text{Gal}(k_2^{(2)}/k)$ and G' its derived group, then it is well known that $C/G' \simeq Cl_2(k)$. An important problem in Number Theory is to determine the structure of G , since the knowledge of G , its structure and its generators solve a lot of problems in number theory as capitulation problems, the finiteness or not of the towers of number fields and the structures of the 2-class groups of the unramified extensions of k within $k_2^{(1)}$. In several times, the knowledge of the rank of G allows to know the structure of G . In this paper, we give an example of this situation.

Let $\mathbf{k} = \mathbb{Q}(\sqrt{pq}, i)$, where p and q are two different primes, then the genus field of \mathbf{k} is $\mathbf{k}^* = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-1})$. According to [7] r_0 , the rank of the 2-class group of \mathbf{k} , is at most equal to 3. Moreover $r_0 = 3$ if and only if $p \equiv q \equiv 1 \pmod{8}$. Let $G = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ be the Galois group of $\mathbf{k}_2^{(2)}/\mathbf{k}$, where $\mathbf{k}_2^{(i+1)}$ is the Hilbert 2-class field of $\mathbf{k}_2^{(i)}$, with $i = 0$ or 1 and $\mathbf{k}_2^{(0)} = \mathbf{k}$. The Artin Reciprocity implies that $r_0 = d(G)$, where $d(G)$ is the rank of G .

2010 *Mathematics Subject Classification.* 11R11, 11R29, 11R32, 11R37.

Key words and phrases. 2-class groups, Hilbert class fields, 2-metacyclic groups.

In [7], the first and the second authors have shown that if $q = 2$, then G is metacyclic non abelian if and only if $p = x^2 + 32y^2$ and $x \not\equiv \pm 1 \pmod{8}$. In this paper, we prove that if p and q are odd different primes, then r , the rank of 2-class group of \mathbf{k}^* , helps to know in which case G is metacyclic.

2. THE RANK OF THE 2-CLASS GROUP OF \mathbf{k}^*

In what follows, we adopt the following notations: If $p \equiv 1 \pmod{8}$ is a prime, then $\left(\frac{2}{p}\right)_4$ will denote the rational biquadratic symbol which is equal to 1 or -1, according as $2^{\frac{p-1}{4}} \equiv \pm 1 \pmod{p}$. Moreover the symbol $\left(\frac{p}{2}\right)_4$ is equal to $(-1)^{\frac{p-1}{8}}$. Let k be a number field and l be a prime; then \mathfrak{l}_k will denote a prime ideal of k above l . We denote, also, by $\left(\frac{x, y}{\mathfrak{l}_k}\right)$ (resp. $\left(\frac{x}{\mathfrak{l}_k}\right)$) the Hilbert symbol (resp. the quadratic residue symbol) for the prime \mathfrak{l}_k applied to (x, y) (resp. x). A 2-group H is said of type $(2^{n_1}, 2^{n_1}, \dots, 2^{n_s})$ if it is isomorphic to $\mathbb{Z}/2^{n_1} \times \mathbb{Z}/2^{n_2} \times \dots \mathbb{Z}/2^{n_s}$, where $n_i \in \mathbb{N}$. For all number field k , $h(k)$ will denote the 2-class number of k . Finally, r_0 (resp. r) denotes the rank of the 2-class group of \mathbf{k} (resp. \mathbf{k}^*).

Lemma 1. *If $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$, then G is cyclic and $r = 1$.*

Proof. If $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$, then, according to [16], the 2-class group of \mathbf{k} is cyclic, so G is an abelian group of rank 1. As \mathbf{k}^* is an unramified extension of \mathbf{k} , then the 2-class group of \mathbf{k}^* is also cyclic and $r = 1$. \square

Lemma 2. *If $p \equiv 1 \pmod{8}$ and $q \equiv 1 \pmod{8}$, then G is a non-metacyclic group.*

Proof. If $p \equiv 1 \pmod{8}$ and $q \equiv 1 \pmod{8}$, then, according to [16], the 2-class group of \mathbf{k} is of rank 3, which is the rank of G . This yields that G is not metacyclic, since the metacyclic groups are of ranks ≤ 2 . \square

According to the two Lemmas 1 and 2, it is interesting to assume, in what follows, that $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$ or $p \equiv 1 \pmod{8}$ and $q \equiv 3 \pmod{4}$. So $r_0 = 2$ (see [16]). We continue with the following lemmas.

Lemma 3 ([11]). *Let $p \equiv 1 \pmod{8}$ be a prime, then*

$$\left(\frac{i}{\mathfrak{p}_{\mathbb{Q}(i)}}\right) = 1 \quad \text{and} \quad \left(\frac{1+i}{\mathfrak{p}_{\mathbb{Q}(i)}}\right) = \left(\frac{2}{p}\right)_4 \left(\frac{p}{2}\right)_4.$$

Lemma 4. *Let $F = \mathbb{Q}(\sqrt{q}, i)$ where $q \equiv 5 \pmod{8}$ and ε_q be the fundamental unite of $\mathbb{Q}(\sqrt{q})$, then*

- (i) $\left(\frac{p, i}{\mathfrak{l}_F}\right) = 1$ for all prime ideal \mathfrak{l}_F of F .
- (ii) $\left(\frac{p, \varepsilon_q}{\mathfrak{l}_F}\right) = 1$ for all odd prime ideal $\mathfrak{l}_F \neq \mathfrak{p}_F$ of F .
- (iii) $\left(\frac{p, \varepsilon_q}{\mathfrak{p}_F}\right) = \left(\frac{p, \varepsilon_q}{2_F}\right) = \begin{cases} 1, & \text{if } \left(\frac{p}{q}\right) = -1; \\ \left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4, & \text{if } \left(\frac{p}{q}\right) = 1. \end{cases}$

Proof. (i) Let \mathfrak{l}_F be an odd prime ideal of F .

If $\mathfrak{l}_F \neq \mathfrak{p}_F$, then \mathfrak{l}_F is a prime ideal of F unramified in $F(\sqrt{q})$ (see the proof of the following theorem), hence

$$\left(\frac{p, i}{\mathfrak{l}_F}\right) = \left(\frac{p}{\mathfrak{l}_F}\right)^{v(i)} = 1, \text{ [12, p. 205].}$$

If $\mathfrak{l}_F = \mathfrak{p}_F$, the prime ideal of F above p , then

$$\begin{aligned} \left(\frac{p, i}{\mathfrak{p}_F}\right) &= \left(\frac{i, p}{\mathfrak{p}_F}\right) \\ &= \left(\frac{i}{\mathfrak{p}_F}\right) && \text{[12, p. 205]} \\ &= \left(\frac{i}{\mathfrak{p}_{\mathbb{Q}(i)}}\right) && \text{[12, p. 205]} \\ &= 1. && \text{(Lemma 3)} \end{aligned}$$

(ii) Same proof as in (i).

(iii) the inertia degree of \mathfrak{p}_F is equal to 1 in $F/\mathbb{Q}(\sqrt{p})$, which implies that

$$\begin{aligned} \left(\frac{p, \varepsilon_q}{\mathfrak{p}_F}\right) &= \left(\frac{\varepsilon_q, p}{\mathfrak{p}_F}\right) \\ &= \left(\frac{\varepsilon_q}{\mathfrak{p}_F}\right) && \text{[12, p. 205]} \\ &= \left(\frac{\varepsilon_q}{\mathfrak{p}_{\mathbb{Q}(\sqrt{q})}}\right) && \text{[12, p. 205]} \end{aligned}$$

Suppose that $\left(\frac{p}{q}\right) = 1$, then

$$= \left(\frac{p}{q}\right)_4 \left(\frac{q}{p}\right)_4. \quad [8, \text{p. 101}]$$

Suppose that $\left(\frac{p}{q}\right) = -1$. With a same argument as above, we get:

$$\left(\frac{p, \varepsilon_q}{\mathfrak{p}_F}\right) = \left(\frac{\mathcal{N}_{\mathbb{Q}(\sqrt{q})/\mathbb{Q}}(\varepsilon_q)}{p}\right) = \left(\frac{-1}{p}\right) = 1, \quad [12, \text{p. 205}].$$

The product formula for the Hilbert symbol implies that $\left(\frac{p, \varepsilon_p}{\mathfrak{p}_F}\right) = \left(\frac{p, \varepsilon_p}{2_F}\right)$. \square

Lemma 5. *Let $F = \mathbb{Q}(\sqrt{q}, i)$ where $q \equiv 3 \pmod{4}$ and ε_q be the fundamental unite of $\mathbb{Q}(\sqrt{q})$. Then*

- (i) $\left(\frac{p, i}{\mathfrak{l}_F}\right) = 1$ for all prime ideal \mathfrak{l}_F of F .
- (ii) $\left(\frac{p, \sqrt{i\varepsilon_q}}{\mathfrak{l}_F}\right) = 1$ for all odd prime ideal $\mathfrak{l}_F \neq \mathfrak{p}_F$ of F .
- (iii) $\left(\frac{p, \sqrt{i\varepsilon_q}}{\mathfrak{p}_F}\right) = \left(\frac{p, \sqrt{i\varepsilon_q}}{2_F}\right) = \begin{cases} 1, & \text{if } \left(\frac{p}{q}\right) = -1; \\ \left(\frac{2}{p}\right)_4 \left(\frac{p}{2}\right)_4 \left(\frac{\sqrt{2\varepsilon_q}}{\mathfrak{p}_{\mathbb{Q}(\sqrt{q})}}\right), & \text{if } \left(\frac{p}{q}\right) = 1. \end{cases}$

Proof. By a similar approach of previous lemma, we get (i) and (ii). For (iii), remark that $2\varepsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$ (see [4]), then $i\varepsilon_q$ is a square in F , since $2\sqrt{i\varepsilon_q} = (1+i)\sqrt{2\varepsilon_q}$. As the Hilbert symbol is a bilinear map with values in

$\{+1, -1\}$ and $2i = (1+i)^2$, so

$$\begin{aligned}
 \left(\frac{p, \sqrt{i\varepsilon_q}}{\mathfrak{p}_F} \right) &= \left(\frac{p, 2}{\mathfrak{p}_F} \right) \left(\frac{p, 1+i}{\mathfrak{p}_F} \right) \left(\frac{p, \sqrt{2\varepsilon_q}}{\mathfrak{p}_F} \right) \\
 &= \left(\frac{p, i}{\mathfrak{p}_F} \right) \left(\frac{p, 1+i}{\mathfrak{p}_F} \right) \left(\frac{p, \sqrt{2\varepsilon_q}}{\mathfrak{p}_F} \right) \\
 &= \left(\frac{1+i}{\mathfrak{p}_F} \right) \left(\frac{\sqrt{2\varepsilon_q}}{\mathfrak{p}_F} \right) \\
 &= \left(\frac{1+i}{\mathfrak{p}_{\mathbb{Q}(i)}} \right) \left(\frac{\sqrt{2\varepsilon_q}}{\mathfrak{p}_{\mathbb{Q}(\sqrt{q})}} \right) [12, \text{p. 205}] \\
 &= \left(\frac{2}{p} \right)_4 \left(\frac{p}{2} \right)_4 \left(\frac{\sqrt{2\varepsilon_q}}{\mathfrak{p}_{\mathbb{Q}(\sqrt{q})}} \right).
 \end{aligned}$$

□

Theorem 1. *Let p and q be primes as above and r be the rank of the 2-class group of $\mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$.*

(1) *If $q \equiv 5 \pmod{8}$, then*

$$r = \begin{cases} 1, & \text{if } \left(\frac{p}{q} \right) = -1; \\ 2, & \text{if } \left(\frac{p}{q} \right) = 1 \text{ and } \left(\frac{p}{q} \right)_4 = -\left(\frac{q}{p} \right)_4; \\ 3, & \text{if } \left(\frac{p}{q} \right) = 1 \text{ and } \left(\frac{p}{q} \right)_4 = \left(\frac{q}{p} \right)_4. \end{cases}$$

(2) *If $q \equiv 3 \pmod{4}$, then, by putting $\eta = \left(\frac{\sqrt{2\varepsilon_q}}{\mathfrak{p}_{\mathbb{Q}(\sqrt{q})}} \right)$ if $\left(\frac{p}{q} \right) = 1$, we obtain*

$$r = \begin{cases} 1, & \text{if } \left(\frac{p}{q} \right) = -1; \\ 2, & \text{if } \left(\frac{p}{q} \right) = 1 \text{ and } \left(\frac{2}{p} \right)_4 = -\left(\frac{p}{2} \right)_4 \eta; \\ 3, & \text{if } \left(\frac{p}{q} \right) = 1 \text{ and } \left(\frac{2}{p} \right)_4 = \left(\frac{p}{2} \right)_4 \eta. \end{cases}$$

Proof. Let F denote the field $\mathbb{Q}(\sqrt{q}, i)$ defined above and ε_q be the fundamental unit of the $\mathbb{Q}(\sqrt{q})$. According to [1], the unit group of F is equal to

$$\begin{cases} \langle i, \varepsilon_q \rangle, & \text{if } q \equiv 5 \pmod{8}; \\ \langle i, \sqrt{i\varepsilon_q} \rangle, & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

As the class number of the F is odd, then by the ambiguous class number formula (see [9]), we have :

$$r = t - e - 1,$$

where t is the number of primes of F that ramify in \mathbf{k}^*/F ($\mathbf{k}^* = \mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$ is the genus field of $\mathbf{k} = \mathbb{Q}(\sqrt{pq}, i)$) and e is determined by $2^e = [E_F : E_F \cap N_{\mathbf{k}^*/F}((\mathbf{k}^*)^\times)]$. The following diagram helps us to calculate the number t .

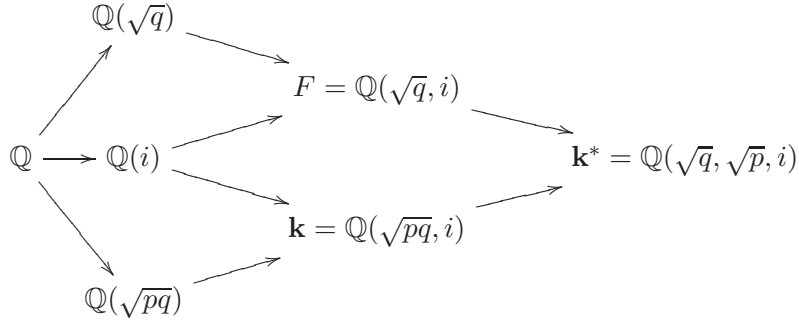


FIGURE 1.

Let l be a prime. Since the extension \mathbf{k}/\mathbf{k}^* is unramified, then

$$e(\mathfrak{l}_F/l) \cdot e(\mathfrak{l}_{\mathbf{k}^*}/\mathfrak{l}_F) = e(\mathfrak{l}_{\mathbf{k}}/l).$$

As

$$e(\mathfrak{l}_F/l) = \begin{cases} 2 & \text{if } l = q \text{ or } 2, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$e(\mathfrak{l}_{\mathbf{k}}/l) = \begin{cases} 2 & \text{if } l = p, q \text{ or } 2, \\ 1 & \text{otherwise,} \end{cases}$$

so it is easy to see that

$$e(\mathfrak{l}_{\mathbf{k}^*}/\mathfrak{l}_F) = \begin{cases} 2 & \text{if } l = p, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, we find that

$$f(\mathfrak{l}_{\mathbf{k}^*}/\mathfrak{l}_F) = \begin{cases} 1 & \text{if } \left(\frac{p}{q}\right) = 1, \\ 2 & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

Therefore

$$t = \begin{cases} 4 & \text{if } \left(\frac{p}{q}\right) = 1, \\ 2 & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

Finally

$$r = \begin{cases} 3 - e & \text{if } \left(\frac{p}{q}\right) = 1, \\ 1 - e & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

The Hasse norm theorem (see e.g. [12, theorem 6.2, p. 179]) implies that a unit ε of F is a norm of an element of $F(\sqrt{p}) = \mathbf{k}^*$ if and only if $\left(\frac{p, \varepsilon}{\mathfrak{p}_F}\right) = 1$, for all $\mathfrak{p}_F \neq 2_F$ prime ideal of F .

If $q \equiv 3 \pmod{4}$, we conclude thanks to the previous lemma that

$$e = \begin{cases} 0, & \text{if } \left(\frac{p}{q}\right) = -1; \\ 1, & \text{if } \left(\frac{p}{q}\right) = 1 \text{ and } \left(\frac{2}{p}\right)_4 = -\left(\frac{p}{2}\right)_4 \eta; \\ 0, & \text{if } \left(\frac{p}{q}\right) = 1 \text{ and } \left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 \eta. \end{cases}$$

This completes the proof of our theorem. \square

3. APPLICATION

Let $G = \text{Gal}(k_2^{(2)}/k)$ be the Galois group of the extension $\mathbf{k}_2^{(2)}/\mathbf{k}$, where $\mathbf{k} = \mathbb{Q}(\sqrt{pq}, i)$ and p, q are distinct primes such that $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$ or $p \equiv 1 \pmod{8}$ and $q \equiv 3 \pmod{4}$. In this section, we give an application of the previous theorem and we characterize the group G . In particular, we will find results about G given by Azizi in [3] and [5].

Theorem 2. *Let p and q be different primes defined as above, $\mathbf{k} = \mathbb{Q}(\sqrt{pq}, i)$, r be the rank of the 2-class group of $\mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$ and $G = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$. Then G is nonmetacyclic if and only if $r = 3$.*

Proof. Since $p \equiv 1 \pmod{8}$, then there exist two integers x and y such that $p = x^2 + 16y^2$. Put $\pi_1 = x + 4yi$, $\pi_2 = x - 4yi$, $\mathbf{k}_1 = \mathbf{k}(\sqrt{\pi_1})$ and $\mathbf{k}_2 = \mathbf{k}(\sqrt{\pi_2})$. As π_1 and π_2 are ramified in $\mathbf{k}/\mathbb{Q}(i)$, then the ideals generated by π_1 and π_2 are squares of ideals of \mathbf{k} . Note that x is odd, thus $x \equiv \pm 1 \equiv i^2 \pmod{4}$,

then the two equations $\pi_i \equiv \xi^2$ are solvable in \mathbf{k} . We conclude that the two extensions $\mathbf{k}(\sqrt{\pi_1})/\mathbf{k}$ and $\mathbf{k}(\sqrt{\pi_2})/\mathbf{k}$ are unramified. It is clear that $\mathbf{k}_i \neq \mathbf{k}^*$. Since $r_0 = d(G) = 2$, then \mathbf{k}_1 , \mathbf{k}_2 and \mathbf{k}^* are precisely the three unramified quadratic extensions of \mathbf{k} . Put $H_i = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}_i)$ where $i = 1, 2$ and $M = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}^*)$. These three subgroups are the maximal subgroups of G . As $d(G) = 2$ and \mathbf{k}_1 is isomorphic to \mathbf{k}_2 , then according to [17], G is nonmetacyclic if and only if $d(M) = r = 3$. \square

Lemma 6. *Let k be an algebraic number field and $G = \text{Gal}(k_2^{(2)}/k)$. Let L be an extension of k such that $M = \text{Gal}(k_2^{(2)}/L)$ is a cyclic subgroup of G of index 2. If four ideal classes of k capitulate in L , then G is abelian or dihedral group.*

Proof. we say that an ideal class of k capitulates in L if it is in the kernel of the homomorphism $j : Cl_2(k) \longrightarrow Cl_2(L)$ induced by extension of ideals from k to L . Furthermore, the homomorphism j corresponds, by the Artin reciprocity law to the group theoretical transfer $V : G/G' \longrightarrow M$ (For further information on V , see for example [15]). Since M is a cyclic subgroup of G of index 2, then, according to [14], G is isomorphic to one of the following groups:

- (1) Cyclic 2-group or 2-group of type $(2^n, 2^m)$.
- (2) The dihedral group.
- (3) The quaternion group.
- (4) The semidihedral group.
- (5) The modular 2-group.

If G is one of the first four groups and four ideal classes of k capitulate in L , then H. Kisilevsky has shown in [13] that G is abelian or dihedral group. Next if G is a modular 2-group, then $G = \langle x, y : x^{2^{n-1}} = y^2, y^{-1}xy = x^{1+2^{n-2}} \rangle$ and $M = \langle x \rangle$. An elementary calculation shows that $\ker(V) = \{G', yG'\}$, i.e. two ideal classes of k capitulate in L . \square

Theorem 3. *Let p and q be different primes and η be the number defined in Theorem 1. Put $\mathbf{k} = \mathbb{Q}(\sqrt{pq}, i)$ and $G = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$.*

- (1) *If $q \equiv 5 \pmod{8}$ and $\left(\frac{p}{q}\right) = -1$, then G is dihedral.*
- (2) *If $q \equiv 5 \pmod{8}$ and $\left(\frac{p}{q}\right) = 1$ and $\left(\frac{p}{q}\right)_4 = -\left(\frac{q}{p}\right)_4$, then G is a non-abelian metacyclic group with G/G' is of type $(2, 4)$.*

- (3) If $q \equiv 3 \pmod{8}$ and $\left(\frac{p}{q}\right) = -1$, then G is abelian or dihedral.
- (4) If $q \equiv 3 \pmod{4}$ and $\left(\frac{p}{q}\right) = 1$ and $\left(\frac{2}{p}\right)_4 = -\left(\frac{p}{2}\right)_4 \eta$, then G is a metacyclic group.

Proof. Proceeding as in the proof of Theorem 7 of [2], we get that if $q \equiv 5 \pmod{8}$, then $h(\mathbf{k}^*) = \frac{h(-p)h(\mathbf{k})}{4}$, where $h(-p)$ denotes the 2-class number of $\mathbb{Q}(\sqrt{-p})$. Since r_0 , the rank of the 2-class group of k , is equal to 2, then G is abelian if and only if $h(\mathbf{k}^*) = \frac{h(\mathbf{k})}{2}$ (see [10]), this is equivalent to $h(-p) = 2$. Which is impossible since $p \equiv 1 \pmod{8}$.

(1) If $q \equiv 5 \pmod{8}$ and $\left(\frac{p}{q}\right) = -1$, then G is nonabelian. According to Theorem 1 the rank of the 2-class group of \mathbf{k}^* is $r = 1$, thus $M = \text{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}^*)$ is cyclic. On the other hand, Azizi in [2] has shown, in this situation, that there are four ideal classes of \mathbf{k} capitulate in \mathbf{k}^* , hence the Lemma 6 implies that G is a dihedral group.

(2) If $q \equiv 5 \pmod{8}$, $\left(\frac{p}{q}\right) = 1$ and $\left(\frac{p}{q}\right)_4 = -\left(\frac{q}{p}\right)_4$, then G is nonabelian and the 2-class group of \mathbf{k} is of type $(2, 4)$ (see [6]). Moreover Theorem 1 yields that $r = 2$, then, according to the previous theorem, we have G is metacyclic.

(3) and (4) are proved similarly. \square

REFERENCES

- [1] A. Azizi, *Unités de certains corps de nombres imaginaires et abéliens sur \mathbb{Q}* , Ann. Sci. Math. Québec **23** (1999), 87-93.
- [2] A. Azizi, *Capitulation of the 2-ideal Classes of $\mathbb{Q}(\sqrt{p_1 p_2}, i)$ Where p_1 and p_2 are primes such that $p_1 \equiv 1 \pmod{8}$, $p_2 \equiv 5 \pmod{8}$ and $\left(\frac{p_1}{p_2}\right) = -1$* , Lecture notes in pure and applied mathematics. vol. **208** (1999), 13-19.
- [3] A. Azizi, *Sur le 2-groupe de classe d'idéaux de $\mathbb{Q}(\sqrt{d}, i)$* , Rend. Circ. Mat. Palermo (2) **48** (1999), 71-92.
- [4] A. Azizi, *Sur la capitulation des 2-classes d'idéaux de $k = \mathbb{Q}(\sqrt{2pq}, i)$, où $p \equiv -q \equiv 1 \pmod{4}$* , Acta. Arith. **94** (2000), 383-399.
- [5] A. Azizi, *Sur une question de Capitulation*, Proc. Amer. Math. Soc. **130** (2002), 2197-2002.
- [6] A. Azizi et M. Taous, *Détermination des corps $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$ dont le 2-groupes de classes est de type $(2, 4)$ ou $(2, 2, 2)$* , Rend. Istit. Mat. Univ. Trieste. **40** (2008), 93-116.
- [7] A. Azizi et M. Taous, *Condition nécessaire et suffisante pour que certain groupe de Galois soit métacyclique*, Ann. Math. Blaise Pascal. **16**, No. 1 (2009), 83-92.

- [8] A. Scholz, *Über die Löbarkeit der Gleichung $t^2 - Du^2 = -4$* , Math. Z. **39** (1934), 95-111.
- [9] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux*, J. Fac. Sc. Tokyo, Sect. 1, t.**2**, (1933), 365-476.
- [10] E. Benjamin, F. Lemmermeyer and C. Snyder, *Real quadratic fields with abelian 2-class field tower*, J. Number Theory. **73** (1998), 182-194.
- [11] F. Lemmermeyer, *Reciprocity Laws*, Springer Monographs in Mathematics, Springer-Verlag. Berlin 2000.
- [12] G. Gras, *Class field theory, from theory to practice*, Springer Verlag 2003.
- [13] H. Kisilevsky, *Number fields with class number congruent to 4 mod 8 and Hilbert's theorem 94*, J. Number Theory **8** (1976), 271-279.
- [14] H. Kurzweil, B. Stellmacher, *The Theory of Finite Groups: An Introduction*, Springer-Verlag. New York 2004.
- [15] K. Miyake, *Algebraic Investigations on Hilbert's Theorem 94, the Principal Ideal theorem and Capitulation Problem*, Expos. Math. **7** (1989), 289-346.
- [16] T. M. McCall, C. J. Parry, R. R. Ranalli, *On imaginary bicyclic biquadratic fields with cyclic 2-class group*, J. Number Theory. **53** (1995), 88-99.
- [17] Y. Berkovich and Z. Janko, *On subgroups of finite p -groups*, Israel J. Math. **171** (2009), 29-49.

ABDELMALEK AZIZI AND ABDELKADER ZEKHNINI: DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES, UNIVERSITÉ MOHAMMED 1, OUJDA, MOROCCO

E-mail address: `abdelmalekazizi@yahoo.fr`

E-mail address: `zekha1@yahoo.fr`

MOHAMMED TAOUS: DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES ET TECHNIQUES, UNIVERSITÉ MOULAY ISMAIL, ERRACHIDIA, MOROCCO

E-mail address: `taousm@hotmail.com`